Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 4, pp.713–725, 1994

0021–8928/94 \$24.00 + 0.00

0021–8928(94)00096-4

THE ANTIPLANE PROBLEM OF A CRACK WITH EDGES TOUCHING PLANES WHERE THE CONSTANTS OF ELASTICITY CHANGE†

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(*Received* 1 *April* 1993)

The antiplane problem of a crack whose edges touch planes where the constants of elasticity change is considered. The special feature of this problem is that it reduces to a singular integral equation whose kernel contains two fixed singularities as well as the traditional movable singularity of the Cauchy kernel. An efficient approximate method of solving equations of this type is presented, based on the construction of a closed solution of a special form of this equation. A method of solving integral equations in the case of a single fixed singularity was presented in [1].

1. STATEMENT OF THE PROBLEM AND ITS REDUCTION TO AN EQUATION WITH TWO FIXED SINGULARITIES

We consider the antiplane problem of the stress concentration around a crack $0<x<1$, $y=0$, $-\infty<z<+\infty$ in a composite elastic space $-\infty<x, y, z<+\infty$, consisting of three perfectly adhering components: a half-space $x<0$, a layer $0<x<1$ and a half-space $x>1$ (with respective moduli of elasticity G_1, G_2 and G_3). Without loss of generality we can assume that a shear load is applied directly to the crack edges, i.e. $\tau_{yz}(x, \pm 0)=-f(x)$, $0<x<1$. The mathematical problem is formulated as the following mixed boundary-value problem for the shear displacement function $w(x,y)$

$$\begin{aligned}
&\Delta w(x, y) = 0, \quad 0<y<+\infty, \quad -\infty<x<+\infty, \quad x\neq 0, \quad x\neq 1\\
&G_1 w_x'(-0,y) = G_2 w_x'(+0,y), \quad 0<y<+\infty\\
&G_2 w_x'(1-0,y) = G_3 w_x'(1+0,y), \quad 0<y<+\infty\\
&w(-0, y) = w(+0, y), \quad w(1-0, y) = w(1+0, y), \quad 0<y<+\infty\\
&w(x, +0) = 0, \quad -\infty<x<0, \quad 1<x<+\infty\\
&G_2 w_y'(x,+0) = -f(x), \quad 0<x<1
\end{aligned} \tag{1.1}$$

To solve problem (1.1) we will use an extended procedure of the method of integral transformations [2]. We put $w(x, +0)=G_2^{-1}\varphi(x)$, $0<x<1$ and apply the Fourier sine-transformation to problem (1.1)

$$w_\alpha(x) = \int_0^\infty \sin\alpha y w(x,y)dy$$

†*Prikl. Mat. Mekh.* Vol. 58, No. 4, pp. 133–145, 1994.

which leads to the following one-dimensional problem

$$w_\alpha''(x)-\alpha^2 w_\alpha(x)=-\alpha G_2^{-1}\varphi(x),\quad -\infty<x<+\infty,\quad x\neq 0,\quad x\neq 1$$
$$(\varphi(x)=0,\quad -\infty<x<0,\quad 1<x<+\infty)$$

$$w_\alpha(-0)=w_\alpha(+0),\quad w_\alpha(1-0)=w_\alpha(1+0) \tag{1.2}$$
$$G_1 w_\alpha'(-0)=G_2 w_\alpha'(+0),\quad G_2 w_\alpha'(1-0)=G_3 w_\alpha'(1+0)$$

Using the procedure [2] for solving discontinuous one-dimensional boundary-value problems, we find that when $0<x,\ \xi<1$ Green's function for problem (1.2) can be represented in the form

$$(-2\alpha)G_\alpha(x,\xi)=e^{-\alpha|x-\xi|}+\Delta_\alpha^{-1}\left(e^{-\alpha(x+\xi)}\cos\pi\alpha_0+e^{-\alpha(2-x-\xi)}\cos\pi\alpha_1+2qe^{-2\alpha}\operatorname{ch}\alpha(x-\xi)\right)$$
$$\Delta_\alpha=1-qe^{-2\alpha},\quad q=\cos\pi\alpha_0\cos\pi\alpha_1 \tag{1.3}$$
$$(G_1+G_2)\cos\pi\alpha_0=G_1-G_2,\quad (G_3+G_2)\cos\pi\alpha_1=G_3-G_2$$

We thus find that

$$G_2 w_\alpha(x)=-\alpha\int_0^1 G_\alpha(x,\xi)\varphi(\xi)d\xi,\quad 0<x<1$$

Inverting the Fourier sine-transform and summing the weakly converging integrals using the expansion

$$\Delta_\alpha^{-1}=\sum_{k=0}^{\infty}q^k e^{-2k\alpha}$$

we obtain $w(x,\ y)$. Imposing the boundary condition $G_2 w_y'(x,\ +0)=-f(x)$ we arrive at the following integral equation for the unknown function $\varphi(x)$

$$L\varphi=-\frac{d}{dx}\int_0^1 L(x,\xi)\varphi(\xi)d\xi=f(x),\quad x\in[0,1]$$
$$\pi L(x,\xi)=\frac{1}{\xi-x}+\frac{\cos\pi\alpha_0}{\xi+x}+\frac{\cos\pi\alpha_1}{\xi+x-2}+R_L(x,\xi) \tag{1.4}$$

$$R_L(x,\xi)=\cos\pi\alpha_0 F(x+\xi)-\cos\pi\alpha_1 F(2-x-\xi)+q\Big[2(x-\xi)\left(4-(x-\xi)^2\right)^{-1}+$$
$$+F(2-x+\xi)-F(2-\xi+x)\Big]$$
$$F(x)=\sum_{k=1}^{\infty}q^k(x+2k)^{-1} \tag{1.5}$$

The aim of the following constructions is to develop an efficient approximate method for solving equations of type (1.4) with $R_L(x,\ \xi)\in C^{(\infty)}([0,\ 1]^2)$. In the specific case (1.5) under discussion $R_L(x,\ \xi)$ is not only infinitely differentiable, but is even analytic. The proposed method is based largely on a previously obtained exact solution for a special case of Eq. (1.4): we first construct a solution of the integral equation

$$\int_0^1 L_0(x,\xi)\varphi(\xi)d\xi=f(x),\quad x\in[0,1]$$

$$\pi L_0(x,\xi)=\frac{1}{\xi-x}+\frac{\cos\pi\alpha_0}{\xi+x}+\frac{\cos\pi\alpha_1}{\xi+x-2}+R_0(x,\xi) \tag{1.6}$$

with a specially chosen regular part $R_0(x, \xi)\in C^{(\infty)}([0, 1]^2)$, and Eq. (1.4) is then solved with $R_L(x, \xi)=R_0(x, \xi)$.

2. CONSTRUCTION OF AN EXACT SOLUTION OF AN INTEGRAL EQUATION WITH TWO FIXED SINGULARITIES

The construction of the exact solution is based on the selection and solution of a pair of mutually dual Riemann–Hilbert problems [3] for a semi-circle.

Let $D=\{z:|z|<1, \ \operatorname{Im} z>0\}$, $\Gamma=\{t:|t|=1, \ 0\leqslant \arg t\leqslant \pi\}$. We denote by $\varphi_1(z)$ and $\varphi_2(z)$ a pair of functions that are analytic in the domain D and which satisfy the following boundary conditions

$$\operatorname{Re}[\varphi_2(t)-\varphi_1(t)]=\operatorname{Im}[\varphi_2(t)+k(t)\varphi_1(t)]=0, \quad -1<t<1$$

$$k(t)=\operatorname{ctg}^2\left(\frac{\pi\alpha_j}{2}\right), \quad -j<t<1-j, \quad j=0,1 \tag{2.1}$$

$$(0<\alpha_j<1, \ j=0,1), \quad \operatorname{Re}\varphi_1(t)=0, \ t\in\Gamma$$

Suppose that the functions $\varphi_j(z)\,(j=1, 2)$ are bounded as $z\to\pm1$ and that as $z\to 0$ we have $|z|^\varepsilon \varphi_j(z)\to 0\,(j=1, 2)$ for every $\varepsilon>0$.

We put

$$\operatorname{Re}\varphi_2(t)=u(t), \quad t\in\Gamma \tag{2.2}$$

$$\operatorname{Im}\varphi_2(t)=-\upsilon(t), \quad t\in\Gamma \tag{2.3}$$

Assuming the function $u(t)$ to be known and $\upsilon(t)$ to be unknown, and then the reverse, we arrive at a mutually dual pair of Riemann–Hilbert [3] problems (2.1), (2.2) and (2.1), (2.3). We assume that the functions $u(t)$, $\upsilon(t)$ satisfy the Hölder condition on $\Gamma(u, \ \upsilon\in H)$ [3] with $u(1)=u(-1)=0$.

We first consider problem (2.1), (2.2). We introduce the notation

$$I=\begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix}, \quad E=\begin{Vmatrix}0 & 1\\ 1 & 0\end{Vmatrix}, \quad A_0=\begin{Vmatrix}1 & -1\\ 1 & -1\end{Vmatrix}, \quad \Omega(z)=\begin{Vmatrix}\varphi_2(z)\\ \varphi_1(z)\end{Vmatrix}, \quad z\in D \tag{2.4}$$

$$N_0(t)=I+n_0(t)A_0, \quad -\infty<t<+\infty$$

$$n_0(t)=\cos\pi\alpha_0, \quad 0<t<+\infty; \quad n_0(t)=\cos\pi\alpha_1, \quad -\infty<t<0$$

Extending the definition of $\Omega(z)$ by the symmetries $\Omega(z)=-\overline{\Omega(\bar z^{-1})}$, $|z|>1$ followed by $\Omega(z)=\overline{E\Omega(\bar z)}$, $\operatorname{Im} z<0$, we arrive at the Riemann boundary-value problem for $\Omega(z)$, the solution of which reduces to the factorization of its matrix coefficient $N_0(t)$ along the real axis ($N_0(t)=I$ when $|t|=1$)

$$N_0(t)=N^+(t)[N^-(t)]^{-1}, \quad -\infty<t<+\infty \tag{2.5}$$

(2.5) is factorized as follows [4]:

$$N(z)=I+\gamma(z)A_0, \quad \gamma(z)=l(z)\cos\pi\alpha_1-l(-z)\cos\pi\alpha_0$$

$$2\pi i l(z)=\ln z, \quad -\pi<\arg z<\pi \tag{2.6}$$

Having solved the above Riemann problem we use the symmetry conditions on $\Omega(z)$ to find that the general solution of the Riemann–Hilbert problem (2.1), (2.2) has the form

$$\begin{Vmatrix}\varphi_2(z)\\ \varphi_1(z)\end{Vmatrix} = ia\begin{Vmatrix}1+2\gamma(z)\\ -1+2\gamma(z)\end{Vmatrix} + \Omega_0(z) \tag{2.7}$$

$$\Omega_0(z) = \frac{1}{2\pi i}\int_\Gamma W(z,\tau)u(\tau)\frac{d\tau}{\tau}$$

$$W(z,\tau) = \left\{\begin{Vmatrix}1\\ 0\end{Vmatrix} + [\gamma(z)-\gamma(\tau)]\begin{Vmatrix}1\\ 1\end{Vmatrix}\right\}\frac{\tau+z}{\tau-z} + \left\{\begin{Vmatrix}0\\ 1\end{Vmatrix} - [\gamma(z)+\gamma(\tau)]\begin{Vmatrix}1\\ 1\end{Vmatrix}\right\}\frac{1+\tau z}{1-\tau z} \tag{2.8}$$

where a is an arbitrary real constant. We find from (2.7) and (2.8) that when $t\in\Gamma$

$$\operatorname{Im}\varphi_2(t) = a[1+2\gamma(t)] - \frac{1}{2\pi}\int_\Gamma \omega(t,\tau)u(\tau)\frac{d\tau}{\tau}$$

$$\omega(t,\tau) = [1+\gamma(t)-\gamma(\tau)]\frac{\tau+t}{\tau-1} - [\gamma(t)+\gamma(\tau)]\frac{1+\tau t}{1-\tau t} \tag{2.9}$$

The kernel $L_0(x, \xi)$ defined by the relation

$$2L_0(x,\xi) = i\omega(\exp(i\pi x),\ \exp(i\pi\xi)) = \operatorname{ctg}\left(\frac{\pi}{2}(\xi-x)\right)[1+\tfrac{1}{2}(\cos\pi\alpha_0-\cos\pi\alpha_1)(\xi-x)] +$$

$$+\tfrac{1}{2}\operatorname{ctg}\left(\frac{\pi}{2}(\xi+x)\right)[(\xi+x)\cos\pi\alpha_1 + (2-\xi-x)\cos\pi\alpha_0] \tag{2.10}$$

can be represented in the form (1.6) with

$$\pi^{-1}R_0(x,\xi) = h_0(\xi-x) + h_1(\xi+x)\cos\pi\alpha_0 - h_1(2-\xi-x)\cos\pi\alpha_1$$

$$h_0(z) = \pi^{-1}q_0 + \tfrac{1}{2}(1+q_0 z)\left(\operatorname{ctg}\frac{\pi z}{2} - \frac{2}{\pi z}\right), \tag{2.11}$$

$$2q_0 = \cos\pi\alpha_0 - \cos\pi\alpha_1,\quad 4h_1(z) = (2-z)\operatorname{ctg}\frac{\pi z}{2} - 2\left(\frac{\pi z}{2}\right)^{-1}$$

where the function $h_j(z)$ is analytic in the strip $-2<\operatorname{Re} z<2j+2,\ j=0, 1$.

Hence, the solution of Eq. (1.6) with $L_0(x, \xi)$ from (2.10) is equivalent to the following integral equation with kernel (2.9)

$$\frac{1}{2\pi}\int_\Gamma \omega(t,\tau)u(\tau)\tau^{-1}d\tau = v(t),\quad t\in\Gamma \tag{2.12}$$

with $\varphi(x) = u(\exp(i\pi x))$, $f(x) = v(\exp(i\pi x))$.

Theorem 1. Let the function $u(t)$, $t\in\Gamma$ be a solution of integral equation (2.12). Then the complex potential $\Omega_0(z)$ from (2.8) is given by the formula

$$\begin{Vmatrix}\varphi_2(z)\\ \varphi_1(z)\end{Vmatrix} = \Omega_0(z),\quad z\in D$$

the solution of boundary-value problem (2.1), (2.3), satisfying the condition

$$\|1,-1\|\lim_{z\to 0} N^{-1}(z)\begin{Vmatrix}\varphi_2(z)\\ \varphi_1(z)\end{Vmatrix}=0 \tag{2.13}$$

with $u(t)=\operatorname{Re}\varphi_2(t),\ t\in\Gamma$.

Suppose that the functions $\varphi_1(z)$, $\varphi_2(z)$ are a solution of boundary-value problem (2.1), (2.3) satisfying condition (2.13). Then $u(t)=\operatorname{Re}\varphi_2(t),\ t\in\Gamma$ is a solution of integral equation (2.12).

Proof. The first part of the theorem without condition (2.13) has already been proved. Condition (2.13) on the potential $\Omega_0(z)$ from (2.8) can be verified directly.

We prove the second part of the theorem. To do this we represent the integral on the left-hand side of (2.12) as follows: $-\operatorname{Im}[\|1,\ 0\|\Omega_0(t)]=(2i)^{-1}\|1,\ 0\|[\Omega_0^+(t)+\Omega_0^-(t)],\ t\in\Gamma$, where $\Omega_0(z)$ is the potential in (2.8). Let $\varphi_1(z)$, $\varphi_2(z)$ be the solution of problem (2.1), (2.3). The definition of the vector $\Omega(z)$ from (2.4) is extended to the rest of the complex plane by the method given above. Here

$$2u(t)=\|1,0\|[\Omega^+(t)-\Omega^-(t)],\ t\in\Gamma$$

which enables us to represent the potential (2.8) as follows:

$$\Omega_0(z)=\frac{1}{2\pi i}N(z)\int_{|t|=1}N^{-1}(t)[\Omega^+(t)-\Omega^-(t)]\frac{(t+z)dt}{2t(t-z)}$$

which is then calculated using the theorem of residues. We finally obtain the difference between the left- and right-hand sides of Eq. (2.12) in the form $C(1+2\gamma(t)),\ t\in\Gamma$ where the pure imaginary constant $2iC$ is the left-hand side of condition (2.13). This proves the second part of the theorem.

We now construct the solution of the Riemann–Hilbert problem (2.1), (2.3). We conformally map the domain D in the z-plane onto the lower half-plane of the s-plane ($\operatorname{Im}s<0$). This mapping is carried out as follows:

$$s=s(z)=\left(\frac{z-1}{z+1}\right)^2,\quad z=z(s)=\frac{1+s^{1/2}}{1-s^{1/2}}\quad (0<\arg s<2\pi)$$

We put

$$\Phi(s)=\begin{Vmatrix}\varphi_1(z(s))\\ \varphi_2(z(s))\end{Vmatrix} \tag{2.14}$$

Symmetrically extending the definition of $\Phi(s)$ to the upper half-plane by

$$\Phi(s)=\operatorname{diag}\{-1,1\}\overline{\Phi(\bar{s})},\quad \operatorname{Im}s>0 \tag{2.15}$$

we arrive at the following Riemann boundary-value problem:

$$\begin{aligned}&\Phi^+(\sigma)=\Phi^-(\sigma)+2iv(z(\sigma))\begin{Vmatrix}0\\ 1\end{Vmatrix},\quad -\infty<\sigma<0\\ &\Phi^+(\sigma)=G(\sigma)\Phi^-(\sigma),\quad 0<\sigma<+\infty\\ &G(\sigma)=A(1-\alpha_0),\quad 0<\sigma<1\\ &G(\sigma)=A(1-\alpha_1),\quad 1<\sigma<+\infty\\ &A(\alpha)=\begin{Vmatrix}\cos\pi\alpha, & -(1+\cos\pi\alpha)\\ 1-\cos\pi\theta, & \cos\pi\alpha\end{Vmatrix}\end{aligned} \tag{2.16}$$

the solution of which is sought in the class of functions that are bounded in a neighbourhood of the points $s=0$, $s=\infty$ and satisfy condition $\Phi(s)|s-1|^{\varepsilon}\to 0$ when $s\to 1$ for all $\varepsilon>0$.

The canonical matrix for solutions [3] of problem (2.16)

$$G(\sigma)=X^{+}(\sigma)[X^{-}(\sigma)]^{-1},\quad 0<\sigma<+\infty$$

is constructed using results obtained by Khvoshchinskaya†

$$X(s)=T(1-\alpha_0)W_0(\alpha_0,\alpha_1;s)\equiv T(1-\alpha_1)W_\infty(\alpha_0,\alpha_1;s)$$

$$T(\alpha)=\operatorname{diag}\left\{\cos\frac{\pi\alpha}{2},\ i\sin\frac{\pi\alpha}{2}\right\}\begin{Vmatrix}1 & 1\\ -1 & 1\end{Vmatrix}$$

$$W_j(\alpha,\beta;s)=\begin{Vmatrix}w_{1,j}(\alpha,\beta;s) & D_s w_{1,j}(\alpha,\beta;s)\\ w_{2,j}(\alpha,\beta;s) & D_s w_{2,j}(\alpha,\beta;s)\end{Vmatrix},\quad j=0,\infty$$

$$D_s w=s(dw/ds)-(1-s)^{-1}w$$

$$w_{1,0}(\alpha,\beta;s)=-w(-(\alpha,\beta;s),\quad w_{2,0}(\alpha,\beta;s)=w(\alpha,\beta;s)$$

$$w_{1,\infty}(\alpha,\beta;s)=w(\beta,\alpha;s^{-1}),\quad w_{2,\infty}(\alpha,\beta;s)=-w(-\beta,\alpha;s^{-1})$$

$$w(\alpha,\beta;s)=\Gamma(\alpha)\Gamma\left(1-\frac{\alpha+\beta}{2}\right)\Gamma\left(1-\frac{\alpha-\beta}{2}\right)(-s)^{(1-\alpha)/2}{}_2F_1\left(1-\frac{\alpha+\beta}{2},\ 1-\frac{\alpha-\beta}{2};\ 1-\alpha;s\right),$$
$$-\pi<\arg(-s)<\pi \tag{2.17}$$

Here $\Gamma(z)$ is the Euler gamma function and ${}_2F_1$ is the Gauss hypergeometric function. The identity in the first of the relations (2.17) is a consequence of the analytic continuation formulae for the corresponding hypergeometric functions and defines two different representations for the same matrix-function $X(s)$.

We need the following notation

$$M(s,\sigma)=\pi^{-3}(2+\alpha_0+\alpha_1)^{-1}(\alpha_0-\alpha_1)^{-1}(\cos\pi\alpha_0-\cos\pi\alpha_1)(D_s-D_\sigma)m_1(s)m_0(\sigma) \tag{2.18}$$

$$m_k(s)=\gamma_k(\alpha_0)[w_{2,0}(\alpha_0,\alpha_1;s)+(-1)^k w_{1,0}(\alpha_0,\alpha_1;s)]\equiv$$

$$\equiv\gamma_k(\alpha_1)[w_{2,\infty}(\alpha_0,\alpha_1;s)+(-1)^k w_{1,\infty}(\alpha_0,\alpha_1;s)]$$

$$\gamma_k(1-\alpha)=\cos\left(\frac{\pi}{2}(k-\alpha)\right),\quad k=0,1$$

where D_s is the differential operator from (2.17).

The partial indices [3] of problem (2.16) are equal to zero. Problem (2.16) has a unique solution. Direct verification shows that this solution satisfies symmetry condition (2.15).

Enforcement of condition (2.13) leads to a single solvability condition for Eq. (2.12). We formulate the final result for Eq. (1.6), (2.10) which is equivalent to (2.12).

Theorem 2. Equation (1.6) with kernel $L_0(x,\ \xi)$ from (2.10) and right-hand side $f(x)\in H$ has a solution in class H if and only if the following solvability condition is satisfied

$$\int_0^1 f(x)m_0\left(-\operatorname{tg}^2\frac{\pi x}{2}\right)\sin^{-1}\pi x\,dx=0 \tag{2.19}$$

†KHVOSHCHINSKAYA L. A., The homogeneous Riemann boundary-value problem for two pairs of functions with piecewise-continuous matrices in the case of two or three singular points. Minsk, 1981. Unpublished paper. Deposited in VINITI, No. 5157-81.

If condition (2.19) is satisfied, then Eq. (1.6), (2.10) has a unique solution in this class which is given by the formulae

$$\varphi(x)=4\int_0^1 M\left(-\mathrm{tg}^2\frac{\pi x}{2},\ -\mathrm{tg}^2\frac{\pi\xi}{2}\right)\frac{f(\xi)\sin^{-1}\pi\xi d\xi}{\cos\pi\xi-\cos\pi x} \tag{2.20}$$

with $\varphi(0)=\varphi(1)=0$.

Let $W_p^{(1)}[0,\ 1]$ be the space of functions $\varphi(x)$ that are absolutely continuous on the segment $[0, 1]$ and for which $\varphi(0)=\varphi(1)=0$, $\varphi'(x)\in L_p(0,\ 1)$ is satisfied with a norm equal to the norm of $\varphi'(x)$ in $L_p(0,\ 1)$, $p>1$.

We consider the equation

$$L_0\varphi=-\frac{d}{dx}\int_0^1 L_0(x,\xi)\varphi(\xi)d\xi=f(x),\quad x\in[0,1] \tag{2.21}$$

Theorem 3. When $1<p<p'=\min\{\alpha_0^{-1},\ \alpha_1^{-1}\}$, the operator $L_0:W_p^{(1)}[0,\ 1]\to L_p(0,\ 1)$ is bounded and has a bounded inverse $L_0^{-1}:L_p(0,\ 1)\to W_p^{(1)}[0,\ 1]$. To prove this we first note that condition (2.19) is not satisfied when $f(x)\equiv 1$ (the integral on the left-hand side of (2.19) is equal to $\frac{1}{2}\pi(\cos(\pi\alpha_0/2)\cos(\pi\alpha_1/2))^{-1}$).

Thus the arbitrary constant which is inherent in the reconstruction of the right-hand side of Eq. (1.6) from the right-hand side of (2.21) is uniquely fixed by condition (2.19). The boundedness of the singular integral operators L_0 and L_0^{-1} in these spaces is verified using appropriate results from [5, 6].

As we show below, by having the exact solution of Eq. (1.6), (2.10) we can take its kernel to be the one in Eq. (1.4). This, as we shall see, is a key point in the method being proposed to solve this equation. The other key point is the construction of efficient formulae for inverting the operator L_0 when its right-hand side is a polynomial, i.e. constructing spectral-type relations in the orthogonal polynomial method [2].

3. THE CONSTRUCTION OF AN EFFICIENT SOLUTION OF THE CHARACTERISTIC EQUATION WITH A POLYNOMIAL RIGHT-HAND SIDE

We solve Eq. (2.21) when the right-hand side is a polynomial. Using formulae from Section 2, we can express the solution in quadratures. However, these formulae are not suitable for numerical implementation. The problem of this section is to obtain representations for such solutions in the form of rapidly converging series.

We consider the polynomial systems $q_n(x)$, $p_n(x)$, $\deg_x p_n=\deg_x q_n=n$ $(n=0,\ 1,\ 2,\ \ldots)$ sequentially, starting with $n=0$. They are defined by the relations

$$\begin{aligned}&q_{n+1}'(x)=-p_n(x),\quad p_n'(x)=q_{n-1}(x)\\&q_0(x)\equiv p_0(x)\equiv 1\quad (q_{-1}(x)\equiv p_{-1}(x)\equiv 0)\end{aligned} \tag{3.1}$$

We construct functions $\varphi_n(x)\,(n=0,\ 1,\ \ldots)$ as solutions of the equations

$$L_0\varphi_n=p_n(x),\quad n=0,1,2,\ldots \tag{3.2}$$

The special choice of polynomials (3.1) for the right-hand side of (3.2) is dictated by the method to be used to construct the functions $\varphi_n(x)$. We note that the choice of polynomials on the right-hand side of (3.2) does not restrict the generality of the construction because the solution for an arbitrary polynomial on the right-hand side of (3.2) can be constructed in an obvious manner from appropriate linear combinations of the functions $\{\varphi_n(x)\}$.

The method of constructing the system of functions $\{\varphi_n(x)\}$ in (3.2) is based on the successive solution of Riemann boundary-value problems

$$\Phi_n^+(\sigma)=\Phi_n^-(\sigma)+2iq_{n+1}(x(\sigma))\begin{Vmatrix}0\\1\end{Vmatrix},\quad -\infty\leqslant\sigma\leqslant 0 \tag{3.3}$$

$$\Phi_n^+(\sigma)=G(\sigma)\Phi_n^-(\sigma),\quad 0\leqslant\sigma\leqslant+\infty,\quad n=-1,0,1,\ldots$$

$$\Psi_n^+(\sigma)=\Psi_n^-(\sigma)+2ip_n(x(\sigma))\begin{Vmatrix}0\\1\end{Vmatrix},\quad -\infty\leqslant\sigma\leqslant 0 \tag{3.4}$$

$$\Psi_n^+(\sigma)=-G(\sigma)\Psi_n^-(\sigma),\quad 0\leqslant\sigma\leqslant+\infty,\quad n=-1,0,1,\ldots$$

in which

$$i\pi x'(s)=s^{-1/2}(1-s)^{-1},\quad 0<\arg s<2\pi$$

$$x(0)=0\ \ (x(\infty)=1),\quad \Phi_{-2}(s)\equiv\Psi_{-1}(s)\equiv 0$$

We shall seek a solution of problems (3.3) and (3.4) in the same class as the solution of problem (2.16). the polynomial $q_{n+1}(x)$ is uniquely defined in terms of the polynomial $p_n(x)$ by the first relation from (3.1) and condition (2.19). The solution of Eq. (3.2) is obtained from the solution of problem (3.3) as follows:

$$2\varphi_n(x)=\|0,1\|\left(\Phi_n^+\left(-\mathrm{tg}^2\frac{\pi x}{2}\right)+\Phi_n^-\left(-\mathrm{tg}^2\frac{\pi x}{2}\right)\right) \tag{3.5}$$

The canonical factorization of the matrix coefficient of problem (3.4)

$$-G(\sigma)=Y^+(\sigma)[Y^-(\sigma)]^{-1},\quad 0<\sigma<+\infty$$

is constructed in the same way as the canonical factorization $X(\sigma)$ of the coefficient $G(\sigma)$ in (2.17) and has the following form

$$Y(s)=\theta(1-\alpha_0)W_0(1-\alpha_0,1-\alpha_1;s)\equiv\theta(1-\alpha_1)W_\infty(1-\alpha_0,1-\alpha_1;s) \tag{3.6}$$

$$\theta(\alpha)=\mathrm{diag}\left\{\cos\frac{\pi\alpha}{2},i\sin\frac{\pi\alpha}{2}\right\}\begin{Vmatrix}1&-1\\1&1\end{Vmatrix}$$

where the matrix-functions $W_j(\alpha,\beta;s)$ ($j=0$, $j=\infty$) are defined in (2.17). The partial indices of the canonical factorization (3.6) are also equal to zero.

The solutions of problems (3.3) and (3.4) are constructed in terms of one another as follows:

$$\Phi_n^\pm(s)=\int_{s_k}^{s}\left[i\sigma^{-1}X(\sigma)a_n-\Psi_n^\pm(\sigma)x'(\sigma)\right]d\sigma+iq_{n+1}(k)\begin{Vmatrix}t_k\\\pm1\end{Vmatrix},\quad k=0,1;$$

$$ia_n=H_X^{-1}\,\mathrm{diag}\{1,-i\}\left\{i\begin{Vmatrix}t_1q_{n+1}(1)-t_0q_{n+1}(0)\\0\end{Vmatrix}+\frac{1}{2}\int_0^{-\infty}\left[\Psi_n^+(\sigma)+\Psi_n^-(\sigma)\right]x'(\sigma)d\sigma\right\}$$

$$n=-1,0,1,2,\ldots \tag{3.7}$$

$$\Psi_n^\pm(s)=\int_{s_k}^{s}\left[i\sigma^{-1}Y(\sigma)b_n+\Phi_{n-2}^\pm(\sigma)x'(\sigma)\right]d\sigma+p_n(k)\begin{Vmatrix}-1\\\pm1\end{Vmatrix},\quad k=0,1;$$

$$ib_n=H_Y^{-1}\,\mathrm{diag}\{1,-i\}\left\{i\begin{Vmatrix}p_n(1)-p_n(0)\\0\end{Vmatrix}-\frac{1}{2}\int_0^{-\infty}\left[\Phi_{n-2}^+(\sigma)+\Phi_{n-2}^-(\sigma)\right]x'(\sigma)d\sigma\right\}$$

$$n = 0, 1, \ldots \tag{3.8}$$

$$\operatorname{diag}\{1, i\} H_{X,Y} = \int_0^{-\infty} X, Y(\sigma)\sigma^{-1} d\sigma$$

$$t_k = \operatorname{tg}^2 \frac{\pi\alpha_k}{2}, \quad k = 0, 1; \quad s_0 = 0, \quad s_1 = \infty \tag{3.9}$$

The numerical matrices H_X, H_Y in (3.9) are non-degenerate, because otherwise the corresponding inhomogeneous problems would have non-trivial solutions in the class under consideration.

The canonical matrix-functions $X(s)$, $Y(s)$ and functions $(-s)^{-1}m_0(s)$, $x'(s)$ have the expansions

$$s^{-1}X(s) = \operatorname{diag}\{1, i\}\rho_k' \sum_{r=0}^{1} \sum_{j=0}^{\infty} \rho_k^{j-(1+\varepsilon_r\alpha_k)/2} X_{k,j}^{(r)}$$

$$s^{-1}Y(s) = \operatorname{diag}\{1, i\}\rho_k' \sum_{r=0}^{1} \sum_{j=0}^{\infty} \rho_k^{j+(-1+\varepsilon_r-\varepsilon_r\alpha_k)/2} Y_{k,j}^{(r)}$$

$$(-s)^{-1} m_0(s) = \rho_k' \sum_{r=0}^{1} \sum_{j=0}^{\infty} \rho_k^{j-(1+\varepsilon_r\alpha_k)/2} \mu_{k,j}^{(r)} \tag{3.10}$$

$$x'(s) = \varepsilon_k \rho_k' \rho_k^{-1/2} \sum_{j=0}^{\infty} l_j \rho_k^j, \quad k = 0, 1; \quad \varepsilon_0 = 1$$

$$\varepsilon_1 = -1, \quad \rho_0 = 1 - \rho_1, \quad \rho_1 = (1-s)^{-1}, \quad \pi l_j = \left(1/2\right)_j (j!)^{-1}$$

which can be obtained using the analytic continuation formulae for the Gauss hypergeometric function ([7], formula 9.131(1)). We note that all series in (3.10) have a radius of convergence equal to unity.

Expansions (3.10) enable us to obtain the following representations for $\Phi_n^\pm(s)$ and $\Psi_n^\pm(s)$

$$\Phi_n^\pm(s) = \operatorname{diag}\{i, 1\} \sum_{r=0}^{1} \sum_{j=0}^{\infty} \rho_k^{j+(1-\varepsilon_r\alpha_k)/2} \lambda_{n,j}^{(k,r)} + i \operatorname{diag}\{1, \pm 1\} \sum_{j=0}^{\infty} \rho_k^{j/2} f_{n,j}^{(k)}, \quad k = 0, 1 \tag{3.11}$$

$$\Psi_n^\pm(s) = \operatorname{diag}\{i, 1\} \sum_{r=0}^{1} \sum_{j=0}^{\infty} \rho_k^{j+(1+\varepsilon_r-\alpha_k\varepsilon_r)/2} æ_{n,j}^{(k,r)} + i \operatorname{diag}\{1, \pm 1\} \sum_{j=0}^{\infty} \rho_k^{j/2} g_{n,j}^{(k)}, \quad k = 0, 1 \tag{3.12}$$

By expressing relations (3.7) using (3.11) and (3.12) we can represent the coefficients of expansions (3.11) in terms of coefficients of expansion (3.12)

$$f_{n,0}^{(k)} = q_{n+1}(k) \begin{Vmatrix} t_k \\ 1 \end{Vmatrix}, \quad \lambda_{n,0}^{(k,0)} = \frac{2}{1-\alpha_k} \operatorname{diag}\{1, -1\} X_{k,0}^{(0)} a_n$$

$$f_{n,2j+1+r}^{(k)} = -\varepsilon_k \left(j(r+1)/2\right)^{-1} \sum_{m=0}^{j} l_{j-m} g_{n,2m+r}^{(k)}$$

$$\nu_{n,j}^{(k,r)} = -\frac{\varepsilon_k}{j+1+\varepsilon_r(1-\alpha_k)/2} \sum_{m=0}^{j} l_{j-m} æ_{n,m}^{(k,r)} \tag{3.13}$$

$$\lambda_{n,j+1-r}^{(k,r)} = \left(j+1+\varepsilon_r(1-\alpha_k)/2\right)^{-1} \operatorname{diag}\{1, -1\} X_{n,j+1-r}^{(r)} a_n + \nu_{n,j}^{(k,r)},$$

$$j = 0, 1, 2, \ldots; \quad r = 0, 1; \quad k = 0, 1$$

$$a_n = -H_X^{-1} \sum_{k=0}^{1} \varepsilon_k \left\{ \left\| \begin{matrix} t_k q_{n+1}(k) \\ 0 \end{matrix} \right\| + \sum_{r=0}^{1} \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j \times \right.$$

$$\left. \times \left[\left(\frac{1}{2}\right)^{\frac{r+1}{2}} \operatorname{diag}\{1,0\} f_{n,2j+1+r}^{(k)} + \left(\frac{1}{2}\right)^{\frac{\varepsilon_r}{2}(1-\alpha_k)+1} \operatorname{diag}\{1,-1\} \nu_{n,j}^{(k,r)} \right] \right\}$$

$$H_X = \sum_{k=0}^{1} \varepsilon_k \sum_{r=0}^{1} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{j+(1-\varepsilon_r\alpha_k)/2}}{j+(1-\varepsilon_r\alpha_k)/2} X_{k,j}^{(r)}$$

There are similar representations (from (3.8)) for the coefficients of the expansions of the $\Psi_n^{\pm}(s)$ in (3.12) in terms of the coefficients of expansions (3.11) for the $\Phi_{n-2}^{\pm}(s)$. From (3.4) these relations enable us to find sequentially the coefficients of expansions (3.11), starting with $n=-1$, and (3.12), starting with $n=1$

$$\left(\Psi_{-1}^{\pm}(s) \equiv 0, \Psi_0^{\pm}(s) \equiv i \left\| \begin{matrix} -1 \\ \pm 1 \end{matrix} \right\| \right).$$

The values of the polynomials $q_{n+1}(x)$ at $x=k$ $(k=0,\ 1)$ which occur in representation (3.13) are uniquely defined by condition (2.19)

$$q_{n+1}(k) = r_n + \varepsilon_k \Delta_{n,0}^{(k)}, \quad k=0,1; \quad \Delta_{n,0}^{(k)} = -\frac{1}{2} \int_0^1 p_n(x)dx$$

$$\Delta_{n,j}^{(k)} = \varepsilon_k \|0,1\| f_{n,j}^{(k)}, \quad j=1,2,\ldots; \quad \pi^2 \left(\cos\frac{\pi\alpha_0}{2} \cos\frac{\pi\alpha_1}{2} \right)^{-1} r_n =$$

$$= \sum_{k=0}^{1} \sum_{r=0}^{1} \sum_{\nu=0}^{1} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{j+(1-\varepsilon_r\alpha_k+\nu)/2}}{j+(1-\varepsilon_r\alpha_k+\nu)/2} \sum_{m=0}^{j} \Delta_{n,2m+\nu}^{(k)} \mu_{k,j-m}^{(r)} \tag{3.14}$$

Here we have used

$$q_{n+1}(x(s)) = \|0,1\| \sum_{j=0}^{\infty} f_{n,j}^{(k)} \rho_k^{j/2} \tag{3.15}$$

The polynomial $p_n(x)$ is obtained from (3.1) in terms of $q_{n-1}(x)$ except, generally speaking, for an arbitrary constant.

In accordance with (3.5) and (3.11) the solutions of Eqs (3.2) $(h=0,\ 1,\ 2,\ \ldots)$ have the following representations

$$\varphi_n(x) = \sum_{r=0}^{1} \left(\sin\frac{\pi x}{2} \right)^{1-\alpha_0\varepsilon_r} \sum_{j=0}^{\infty} \varphi_{n,j}^{(0,r)} \sin^{2j}\frac{\pi x}{2} =$$

$$= \sum_{r=0}^{1} \left(\cos\frac{\pi x}{2} \right)^{1-\alpha_1\varepsilon_r} \sum_{j=0}^{\infty} \varphi_{n,j}^{(1,r)} \cos^{2j}\frac{\pi x}{2} \tag{3.16}$$

$$\varphi_{n,j}^{(k,r)} = \|0,1\| \lambda_{n,j}^{(k,r)}, \quad j=0,1,2\ldots; \quad k,r=0,1$$

where the series in (3.15) and (3.16) have unit radii of convergence, just like (3.11) and (3.12).

4. CONSTRUCTION OF AN APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION OF THE PROBLEM

The preceding sections enable us to proceed directly to the approximate solution of integral equation (1.4). All constructions will be performed subject to the condition

$$L\varphi = 0, \quad \varphi \in W_p^{(1)}[0,1], \quad p>1 \Rightarrow \varphi = 0 \tag{4.1}$$

This condition will be satisfied if the operator L is strictly positive, i.e.

$$(L\varphi,\varphi)>0, \quad \varphi \neq 0, \quad \varphi \in W_p^{(1)}[0,1], \quad p>1$$

$$(\varphi,f) = \int_0^1 \varphi(x)f(x)dx \tag{4.2}$$

This is in fact the case for the operator L in Eq. (1.4), (1.5) because $(L\varphi, \varphi)$ is the energy integral for problem (1.1) in which the boundary condition $G_2w_y'(x, 0)=-f(x)$, $x\in[0, 1]$ is replaced by $w(x, 0)=G_2^{-1}\varphi(x)$.

We will describe the scheme for the approximate solution of Eq. (1.4) applied to Eq. (1.4), (1.5).

Later constructions are based on the representation

$$(L-L_0)\varphi = R\varphi = \int_0^1 R(x,\xi)\varphi(\xi)d\xi \tag{4.3}$$

where L_0 is the operator from (2.21), together with the subsequent equivalent regularization

$$\varphi + L_0^{-1}R\varphi = L_0^{-1}f \tag{4.4}$$

In the case of (1.5) the kernel $R(x, \xi)$ in (4.3) can be represented (see also (2.11)) by a Taylor series

$$\begin{aligned} &\pi R(x,\xi) = \sum_{j=1}^{\infty} \beta_j(x,\xi), \quad j^{-1}\beta_j(x,\xi) = \beta_j^{(0)}\left(\tfrac{1}{2}\right)^j(\xi-x)^{j-1} - \beta_j^{(1)}(\cos\pi\alpha_0 - \\ &-(-1)^j\cos\pi\alpha_1)(1-\xi-x)^{j-1}, \quad j=1,2,\ldots \\ &\beta_{2m-s}^{(0)} = (2q_0)^{1-s}\xi(2m) - sq(1+2q\Phi(q,2m-s,2)) \\ &\beta_{2m-s}^{(1)} = (1-2^{-2m})\xi(2m) - 1 + q2^{s-2m}\Phi\left(q,2m-s,\tfrac{3}{2}\right), \quad m=1,2,\ldots; \quad s=0,1 \end{aligned} \tag{4.5}$$

where the functions ξ and Φ are given by formulae 9.522(1) and 9.550 from [7]. The rate of convergence of series (4.5) for x, $\xi\in[0, 1]$ is given by the relations

$$\overline{\lim_{j\to\infty}} \left|\beta_j^{(k)}\right|^{1/j} = (2k+1)^{-1}, \quad k=0,1 \tag{4.6}$$

Equation (4.4) is a Fredholm equation of the second kind. It can be shown (see (4.1), (4.5) and (4.6)) that it has a unique solution in $W_p^{(1)}[0, 1]$ $(1<p<p')$ [8]. The following constructions are justified by the general theory of approximate methods for solving equations of the second kind [8].

Replacing the kernel $R(x, \xi)$ by a partial sum of series (4.5)

$$\pi R_n(x,\xi) = \sum_{j=1}^{n+1} \beta_j(x,\xi), \quad R_n\varphi = \int_0^1 R_n(x,\xi)\varphi(\xi)d\xi$$

we construct an approximate solution θ_n of Eq. (4.4) (and hence of (1.4), (1.5)) as a solution to the following equation with degenerate kernel

$$\theta_n + L_0^{-1}R_n\theta_n = L_0^{-1}f \tag{4.7}$$

having first transformed the kernel to the form

$$R_n(x,\xi) = \sum_{k=0}^{n}\sum_{j=1}^{n} r_{kj}^{(n)} p_k(x) p_j(\xi)$$

(the polynomials $p_k(x)$ $(k = 0, 1, 2, \ldots)$ are defined in Section 3).

We seek a solution of Eq. (4.7) in the form

$$\theta_n(x) = L_0^{-1}f + \sum_{m=0}^{n} \theta_m^{(n)} \varphi_m(x)$$

(the functions $\varphi_j(x)$ $(j = 0, 1, 2, \ldots)$ are defined in Section 3 ((3.1), (3.11) and (3.16)). Proceeding as in [9], we find the following finite system of linear algebraic equations for the unknown coefficients of this representation

$$\theta_k^{(n)} + \sum_{m=0}^{n} a_{km}^{(n)}\theta_m^{(n)} = b_k^{(n)}, \quad k = 0,1,\ldots,n$$

$$a_{km}^{(n)} = \sum_{j=0}^{n} r_{kj}^{(n)}(\varphi_m, p_j), \quad b_k^{(n)} = \sum_{j=0}^{n} r_{kj}^{(n)}(L_0^{-1}f, p_j)$$

To find the matrix elements of this system it is necessary to evaluate integrals of the form $(\varphi_n,\ p_m)$. To do this we make a change of variables $x = x(s)$. Using (3.1) we obtain

$$(\varphi_n, p_m) = -\int_0^{-\infty} \varphi_n(x(s)) \frac{d}{ds} q_{m+1}(x(s)) ds$$

Splitting the range of integration into two, from 0 to -1 and from -1 to $-\infty$, and then using the appropriate local expansions from (3.14)–(3.16), we find that

$$(\varphi_n, p_m) = -\gamma_{n,m}\left(\tfrac{1}{2}\right)$$

$$\gamma_{n,m}(z) = \frac{1}{2}\sum_{k=0}^{1}\sum_{r=0}^{1}\sum_{\nu=0}^{1}\sum_{j=0}^{\infty} \frac{z^{j+1+(\nu-\varepsilon_r\alpha_k)/2}}{j+1+(\nu-\varepsilon_r\alpha_k)/2} \sum_{\delta=0}^{j} (2\delta+\nu+1)\Delta_{m,2\delta+\nu+1}^{(k)} \varphi_{n,j-\delta}^{(k,r)}$$

where, as in (3.15), (3.16), (3.11) and (3.12), the radius of convergence of the series $\gamma_{n,m}(z)$ is equal to unity.

The rate of convergence of the approximate solution constructed in this manner is given by the estimate

$$\|\varphi - \theta_n\|_{W_p^{(1)}} \le \delta_n, \quad \delta_n = \delta 2^{-n},$$

which follows from (4.6).

In the more general case when $R_L(x,\ \xi) \in C^{(\infty)}([0,\ 1]^2)$ and condition (4.1) is satisfied, the solution is constructed using the same procedure. If the approximation to $R_n(x,\ \xi)$ is taken in the form of a partial sum of an appropriate series of classical orthogonal polynomials, as in the example

$$R_n(x,\xi) = \sum_{k=0}^{n}\sum_{j=0}^{n} \beta_{kj} T_k(1-2x) T_j(1-2\xi)$$

(where $T_k(x)$ are Chebyshev polynomials of the first kind and β_{kj} are Fourier coefficients of the double series expansion of $R(x, \xi)$ for the given polynomials), the sequence of estimates for the rate of convergence δ_n will satisfy the property that for every natural number m the sequence $\{\delta_n n^m, \ n=1, 2, \ldots\}$ is bounded.

All the constructions of this paper were performed subject to the condition

$$|\cos\pi\alpha_0|<1, \quad |\cos\pi\alpha_1|<1 \tag{4.8}$$

These restrictions can be made less rigorous only one of the inequalities (4.8) need be strict. The procedure for solving Eq. (1.6), (2.10) for the four exclusive cases remains unchanged. The only change is in the formulae for factorizing the matrix-functions $G(s)$, which is performed as in the previously-cited paper by Khvoshchinskaya.

REFERENCES

1. ONISHCHUK O. V., On a method of solving integral equations and its application to the problem of the bending of a plate with a cross-shaped inclusion. *Prikl. Mat. Mekh.* **52**, 2, 269–283, 1988.
2. POPOV G. Ya., *Elastic Stress Concentrations Around Dies, Cuts, Thin Inclusions and Supports.* Nauka, Moscow, 1982.
3. MUSKHELISHVILI, N. I., *Singular Integral Equations*. Nauka, Moscow, 1968.
4. KHRAPKOV A. A., Some cases of the elastic equilibrium of an infinite wedge with an asymmetric cut at the vertex under the action of point forces. *Prikl. Mat. Mekh.* **35**, 4, 677–689, 1971.
5. KHVEDELIDZE B. V., Linear discontinuous boundary-value problems in the theory of functions and singular integral equations, and some of their applications. *Tr. Tbil. Mat. Inst. im. A. M. Razmadze* **23**, 3–158, 1957.
6. DUDUCHAVA R. V., Integral convolution equations with discontinuous presymbols, singular integral equations with fixed singularities, and their applications to mechanical problems. *Tr. Tbil. Mat. Inst. im. A. M. Razmadze* **60**, 3–135, 1979.
7. GRADSHTEIN I. S. and RYZHIK I. M., *Tables of Integrals, Sums, Series and Products*. Nauka, Moscow, 1971.
8. KANTOROVICH L. V. and AKILOV G. P., *Functional Analysis.* Nauka, Moscow, 1984.
9. MIKHLIN S.G., *Integral Equations and Their Application to some Problems in Mechanics, Mathematical Physics and Technology*. Gostekhizdat, Moscow, 1949.

Translated by R.L.Z.